Lecture 17

Countability (contd.) and Relations

An Uncountable Set

We will prove now that $\mathbb{R}$, the set of real numbers, is an uncountable set.

Suppose $\mathbb{R}$ is countably infinite, then X, the set of real numbers in $(0,1)$, is also countable.

Because subsets of countable sets are also countable.

X is also an infinite set. (Why?)

Then, elements of X can be listed in some order, say, $r_1, r_2, r_3, \ldots,$

$$
\begin{aligned}
r_1 &= 0\,.\,d_{11}\,d_{12}\,d_{13}\,d_{14}\,d_{15}\ldots \\
r_2 &= 0\,.\,d_{21}\,d_{22}\,d_{23}\,d_{24}\,d_{25}\ldots \\
r_3 &= 0\,.\,d_{31}\,d_{32}\,d_{33}\,d_{34}\,d_{35}\ldots \\
r_4 &= 0\,.\,d_{41}\,d_{42}\,d_{43}\,d_{44}\,d_{45}\ldots \\
r_5 &= 0\,.\,d_{51}\,d_{52}\,d_{53}\,d_{54}\,d_{55}\ldots \\
&\vdots
\end{aligned}
$$

where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

An Uncountable Set

Construct number $r = 0.d_1d_2d_3d_4d_5\ldots$, such that:

$$d_i = \begin{cases} 0, \text{ if } d_{ii} \neq 0 \\ 1, \text{ if } d_{ii} = 0 \end{cases}$$

r **cannot exist** in the sequence $r_1, r_2, r_3, \ldots$, because:

- r differs from r_1 on d_{11}.
- r differs from r_2 on d_{22}.
- $\ldots$
- r differs from r_k on d_{kk}.

Reached a contradiction. Hence, $\mathbb{R}$ is uncountable.

$$\begin{aligned}
r_1 &= 0\,.\,d_{11}\,d_{12}\,d_{13}\,d_{14}\,d_{15}\ldots \\
r_2 &= 0\,.\,d_{21}\,d_{22}\,d_{23}\,d_{24}\,d_{25}\ldots \\
r_3 &= 0\,.\,d_{31}\,d_{32}\,d_{33}\,d_{34}\,d_{35}\ldots \\
r_4 &= 0\,.\,d_{41}\,d_{42}\,d_{43}\,d_{44}\,d_{45}\ldots \\
r_5 &= 0\,.\,d_{51}\,d_{52}\,d_{53}\,d_{54}\,d_{55}\ldots \\
&\vdots
\end{aligned}$$

This proof method is called as Cantor's diagonalization argument.

■

Schröder-Bernstein Theorem

Schröder-Bernstein Theorem: If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are injunctions from A to B and from B to A, then there is a bijection from A to B.

Proof: Skipped because it is not easy.

Using Schröder-Bernstein Theorem

Example: Show that $|A| = |B|$, where $A = (0,1)$ and $B = (0,1]$.

Solution: Giving bijection is not easy so we will use Schröder-Bernstein Theorem.

Injunction f_1 from A to B:

$$f_1(n) = n$$

Injunction f_2 from B to A:

$$f_2(n) = n/2$$

Proving they are injunction is trivial.

Since, f_1 is an injunction from A to B and f_2 is an injunction from B to A, there is a bijection between A and B. Thus, $|A| = |B|$. ■

The Continuum Hypothesis

The Continuum Hypothesis: There is no set A, such that $|\mathbb{Z}| < |A| < |\mathbb{R}|$.

It was the first problem in the list of 23 open problems proposed by Hilbert in 1900.

It's proven that it can be neither proved nor disproved under ZFC or ZF axioms.

Relations

A **relation** is a way to relate the elements of two (not necessarily different) sets.

Definition: Let A and B be sets. A **binary relation**, say R, from A to B is a subset of $A \times B$. We use aRb to denote $(a, b) \in R$ and $a\not R b$ to denote $(a, b) \notin R$.

Example: Let A be the set of cities in India, and let B be the set of states in India.
We can define a relation R so that $(a, b) \in R$, if city a is in state b.
For instance, *(Jodhpur, Rajasthan), (Kanpur, U.P.), (Chennai, Tamilnadu)* are in R.

Note: A function can be seen as a relation, but a relation is not necessarily a function.

Definition: A **relation on a set** A is a relation from A to A.

Example: Let $A = \{1,2,3,4\}$ and $R = \{(a, b) \mid a \text{ divides } b\}$.

Then, $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Properties of Relations

Definition: A relation R on a set A is called **reflexive** if $(a, a) \in R$, $\forall a \in A$.

Definition: A relation R on a set A is called **symmetric** if $(a, b) \in R$ implies $(b, a) \in R$, $\forall a, b \in A$.

Definition: A relation R on a set A is called **antisymmetric** if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$, $\forall a, b \in A$.

Note: A relation can have or lack both of symmetric and antisymmetric properties.

Definition: A relation R on a set A is called **transitive** if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$, $\forall a, b, c \in A$.